UDC 521.1

CERTAIN PROPERTIES OF D'ALAMBERT FUNCTIONS OF CELESTIAL MECHANICS

PMM Vol.41, №5, 1977, pp. 870-875
K. V. KHOLSHEVNIKOV
(Leningrad)
(Received March 30, 1977)

It is shown that the d'Alambert properties of functions of celestial mechanics are the corollary of holomorphy is an unbounded region and of some functional relationship. The radius of convergence of related power expansions is determined by the region size.

In the construction of theories of motion of planets and satellites considerable use is made of the d'Alambert characteristic (see [1]), which is understood to be the property of coefficients of trigonometric series which makes it possible to substitute for it power series $x = re^{i\varphi}$ and $y = re^{-i\varphi}$, where φ is the angle variable and r is the eccentricity or declination. It is reasonable to pose the problem of determination of the functional properties of the sum, which have as a consequence the indicated characteristic of coefficients. As far as the author is aware no attempts were made to solve that problem, apparently owing to the following causes. First, in the case of the usual in practice formal constructions (without convergence test and estimate of the reminder) it is sufficient to formally determine the d'Alambert characteristics, without excluding even series that are everywhere divergent. Second, the standard region of variation of variables r and φ , which is the Cartesian product of circle $|r| < R$ by the band $|\operatorname{Im} \varphi| < \Phi$ is not suitable for analyzing d'Alambert properties. In fact, in variables x, y the coordinate origin $x = y = 0$ is on the boundary, since any sphere as small as desired $|x|^2 + |y|^2 < \varepsilon$ contains points with arbitrary large $|\operatorname{Im} \varphi|$.

In the present paper a method is presented for reducing the d'Alambert property to a certain functional property. The key feature is the determination of a suitable region of variation of variables r and φ. As an example of application of this method, estimates of the common term and of the remainder of the related series, are obtained.

Investigation of the d'Alambert characteristic directly affects the question of correlating Cartesian phase coordinates x, y and polar r, φ in various problems of dynamics, as for instance in the problem of solving a system of differential equations in the equilibrium neighbourhood of periodic or quasi-periodic motion. Depending on the problem features and preference of the researcher Cartesian or polar coordinates are used. Formal solutions are usually the same, but the evaluation of applicability regions can be substentially different.

The investigations presented in this paper imply the isomorphism of solutions and related regions in both coordinate systems, when certain conditions, conveniently called d'Alambert conditions, are satisfied. Obtained results can be applied to numerous problems, for instance, of estimating coefficients of expansion of perturbing functions in the problem of several bodies (*). More interesting is the application to the problem of quasi-periodic solutions of the canonical system of equations with the Hamiltonian $H_0(p)+\mu H_1(p,q)$ (in generally accepted notation) in the case of limit degeneration of H_0 typical for problems of celestial mechanics [2]. The sequence of contracting regions [2] in which is determined the canonical system

$$H_{0k}(p_k)+\mu_k H_{1k}(p_k, q_k) \quad (\mu_k \to 0)$$

after the k-th substitution of variables, may be modified so that $r=0$ is an internal point that does not protrude at the first step. Here r corresponds to the variable or group of variables p with which the limit degeneration takes place.

The determination of the estimate of the perturbing function or the analysis of contracting regions is too cumbersome to be used as an example for the theorems of this work. Two simple examples will be given instead. The second of these shows that the extension of derived results to the case of more than two dimensions is hardly of any interest; in problems of dynamics the variables that show the d'Alambert characteristic is usually of two kinds: action and angle.

Let r and φ vary in the multiply connected region D_1 of space C^2, defined by the inequality

$$0<|r|e^{|v|}<R \quad (\varphi = u+iv)$$

where R is an arbitrary positive number. Region D_1 differs from the simply connected region D_2: $|r|e^{|v|}<R$ by the absence of the plane $r=0$. It can be shown that the mapping of D_1 by the entire functions

$$x = re^{i\varphi}, \quad y = re^{-i\varphi} \tag{1}$$

on E_1: $0<|x|<R,\ 0<|y|<R$ is multi-sheeted. The conventional identification $(r,\varphi)\sim(r,\varphi+2\pi)$ results in a two-sheeted mapping. The identification

$$(r,\varphi)\sim(-r,\varphi+\pi) \tag{2}$$

intrinsic to polar coordinates, implies univalence.

Thus the entire functions (1) map one-to-one D_1° onto E_1. Here D_1° is the factor set of D_1 with respect to the equivalence (2).

We call f bounded d'Alambert R- function, if it is holomorphic, and bounded in D_1° and satisfies the relationship

$$f(r,\varphi) = f(-r,\varphi+\pi) \tag{3}$$

Region D_1 differs from the singly connected region D_2 by the fine set $r=0$. By the theorem on the obliteration of singularities (see [3]) the d'Alambert function is

(*) Editor's note. Commonly referred to as the three body problem or the n body problem.

holomorphic and bounded in D_2.

The composition $f^* (x, y) = f (r (x, y), \varphi (x, y))$ is obviously holomorphic and bounded in E_1 by the same theorem on the whole bicircle E_2: $x | < R$, $| y | < R$.

Conversely, let f^* be holomorphic and bounded in E_1 and consequently, also, on E_2. By the substitution of variables (1) transforms it into a bounded d'Alambert R-function in D_1° and consequently in D_2°. It is evidently sufficient to specify boundedness only in the neighbourhood of singular sets $r = 0$ or $xy = 0$ (here singularity means violation of the one-to-one mapping of (1)). We call a function d'Alambert R- function if it is a d'Alambert R'-function for an $R' < R$.

The fundamental result is formulated as follows: the substitution of variables (1) transforms the d'Alambert R-function f into function f^* holomorphic on the bicircle E_2, and conversely.

Let us investigate some of properties of the d'Alambert R- functions.

Owing to its periodicity function f expands into Fourier series

$$f (r, \varphi) = \sum_{-\infty}^{\infty} c_n (r) e^{in\varphi} \tag{4}$$

whose half-width convergence depends on r as

$$| v | < \ln | R / r | \tag{5}$$

which implies that for bounded functions $c_n = O (| r |^{|n|})$. We denote

$$M = \sup_{\substack{|r|<R \\ v=0}} | f (r, \varphi) |, \quad M_1 = \sup_{|r|<R} \frac{1}{2\pi} \int_0^{2\pi} | f (r, \varphi) | d\varphi \tag{6}$$

$$M_2^2 = \sup_{|r|<R} \frac{1}{2\pi} \int_0^{2\pi} | f (r, \varphi) |^2 d\varphi, \quad M_1 \leqslant M_2 \leqslant M$$

Since c_n can be represented by an integral along the real axis, c_n are holomorphic in circle $| r | < R$ and bounded by the number M_1.

L e m m a. If function g is holomorphic in circle $| r | < R$, bounded by number G, and when $r \to 0$ is of order $| r |^n$, then the exact estimate

$$| g (r) | \leqslant G | r / R |^n \tag{7}$$

is valid in that circle.

In fact, if the inequality is postulated for $n = 0$, it is transformed into the Schwartz lemma when $n = 1$ (see [3]). Extension to an arbitrary n is trivial, hence it is possible to call the formulated statement the Schwartz lemma independently of the values of n.

The obtained properties of coefficients c_n and the Schwartz lemma imply that

$$| c_n (r) | \leqslant M_1 | r / R |^{|n|} \tag{8}$$

from which follows the analog of the Cauchy-Hadamard formula

$$\overline{\lim_{n\to\infty}} \sqrt[|n|]{|c_n(r)|} \leqslant |r/R| \tag{9}$$

Relationship (3) implies that $c_n(r) = (-1)^n c_n(-r)$, hence with allowance for (8), we have

$$c_n(r) = (r/R)^{|n|}\, \sigma_n(r^2) \tag{10}$$

where functions σ_n in the circle $|r| < R$ are holomorphic and bounded by the number M_1.

Let us estimate the totality of coefficients with symmetric indices. We fix n and r_0 and denote

$$c_n(r_0) = |c_n(r_0)|\, e^{i\psi_n}$$
$$B(r) = c_n(r)e^{-i\psi_n} + c_{-n}(r)e^{-i\psi_{-n}}$$

Evidently

$$B(r) = \frac{1}{2\pi}\int_0^{2\pi} \zeta f(r, \varphi)\, d\varphi, \quad \zeta = e^{i(-n\varphi-\psi_n)} + e^{i(n\varphi-\psi_{-n})}$$

By the Buniakowski inequality

$$|B(r)| \leqslant \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(r, \varphi)|^2\, d\varphi\right\}^{1/2} \left\{\frac{1}{2\pi}\int_0^{2\pi} |\zeta|^2\, d\varphi\right\}^{1/2}$$

Since $|\zeta|^2 = 2 + 2\cos(2n\varphi + \psi_n - \psi_{-n})$, hence for $n \neq 0$ we have $|B(r)| \leqslant M_2\sqrt{2}$ and by the Schwartz lemma $|B(r)| \leqslant M_2\sqrt{2}\, |r/R|^{|n|}$. Substituting $r = r_0$ and omitting the subscript by virtue of the arbitrariness of r_0, we finally obtain

$$|c_n(r)| + |c_{-n}(r)| \leqslant M_2\sqrt{2}\, |r/R|^{|n|} \quad (n \neq 0) \tag{11}$$

Similarly we can obtain

$$|c_n(r)| + |c_{-n}(r)| < 4M\pi^{-1}\, |r/R|^{|n|} \quad (n \neq 0) \tag{12}$$

Owing to the holomorphy of function f with respect to r, it expands into the Maclaurin series

$$f(r, \varphi) = \sum_{k=0}^{\infty} a_k(\varphi)\, r^k \tag{13}$$

whose radius of convergence depends on v:

$$|r| < Re^{-|v|} \tag{14}$$

The integral Cauchy formulas determine coefficients a_k for any φ. The corresponding Cauchy estimate yields for bounded functions

$$|a_k(\varphi)| \leqslant M_3(v)\left(\frac{e^{|v|}}{R}\right)^k \tag{15}$$

$$2\pi M_3(v) = \sup \int_0^{2\pi} |f(R'e^{iz}, u + iv)|\, dz \quad (-\infty < u < +\infty, 0 < R' < Re^{-|v|})$$

Note that $M_3(v) \leqslant M_0 \equiv \sup_{D_2} |f(r, \varphi)|$. In this case $M_0 = M$, since the cylinder $|r| = R$ and $v = 0$ is the Shilov boundary of region D_2 (see [3]).

Formula (15) shows that $a_k(\varphi)$ is a trigonometric polynominal of power not higher than k. It follows from (3) that

$$a_k(\varphi + \pi) = (-1)^k a_k(\varphi) \tag{16}$$

Hence

$$a_k(\varphi) = \sum_{m=0}^{k} a_{km} e^{i(k-2m)\varphi} \tag{17}$$

Relationship (17) was proved by another method in [4].

Properties (8)-(17) extend to arbitrary d'Alambert R- functions. For instance, formula (8) is trivial when $M_1 = \infty$; when $M_1 < \infty$ it is established by passing to limit $R' \to R$. To obtain an effective inequality when $M_1 = \infty$ it is necessary to substitute the smaller number R' for R and in formula (6), which determines M_1, takes the upper bound for $|r| < R'$.

Similar constructions can be obtained with elements $x' = r\cos\varphi$ and $y' = r\sin\varphi$. In that case regions D_1 and D_2 are defined by the inequalities

$$0 < |r| \sqrt{\operatorname{ch} 2v} < R, \quad |r| \sqrt{\operatorname{ch} 2v} < R$$

Region E_1 is then a sphere without two planes $|x'|^2 + |y'|^2 < R^2$, $x'^2 + y'^2 \neq 0$, and region E_2 is the sphere $|x'|^2 + |y'|^2 < R^2$. The region of convergence of series (4) and (13) increases

$$2|v| < \ln \frac{R^2 + \sqrt{R^4 - |r|^4}}{|r|^2}, \quad |r| < \frac{R}{\sqrt{\operatorname{ch} 2v}}$$

Hence relationships (4)-(17) remain valid, and the estimates may be somewhat strengthened. For instance the inequality (15) may be replaced by

$$|a_k(\varphi)| \leqslant M_3(v)\left(\frac{\sqrt{\operatorname{ch} 2v}}{R}\right)^k$$

The following are the simplest d'Alambert functions: 1, r^2, polynominal in r^2, and function r^2, $r^{|n|} e^{in\varphi}$ which is holonomic in the origin neighbourhood. (The simplest non-d'Alambert functions are: r, $\cos n\varphi$.) D'Alambert functions comprise the sum, the difference, the quotient (if the denominator has no roots in D_2°), and the product of d'Alambert functions; the sum of d'Alambert function series locally convergent in D_2°, and a function of several d'Alambert functions, which is holomorphic in related regions.

Let us consider the operators

$$A_1 f = r\frac{\partial f}{\partial r}, \quad A_2 f = \frac{\partial f}{\partial \varphi}$$

$$A_3(f, g) = \frac{1}{r}(f, g)_{r\varphi} \equiv \frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \varphi}\frac{\partial g}{\partial r}\right)$$

$$A_4 f(r, \varphi) = \frac{1}{r}\int\limits_0^r f(z, \varphi)\, dz, \quad A_5 f(r^2) = \frac{1}{2\pi}\int\limits_0^{2\pi} f(r, \varphi)\, d\varphi$$

$$A_6 f(r, \varphi) = \int [f(r, \varphi) - A_5 f(r^2)]\, d\varphi$$

where the constant in the indefinite integral is selected so as to satisfy condition $A_5(A_6 f) = 0$. Operators A_k transform the set of d'Alambert R- functions into itself. For A_1, A_2, and A_3 this is implied by the equalities

$$r\frac{\partial}{\partial r} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = x'\frac{\partial}{\partial x'} + y'\frac{\partial}{\partial y'}$$

$$\frac{\partial}{\partial \varphi} = i\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right) = -y'\frac{\partial}{\partial x'} + x'\frac{\partial}{\partial y'}$$

$$(fg)_{r\varphi} = -2ir(fg)_{xy} = r(fg)_{x'y'}$$

For A_4 and A_5 this follows from their limit $\max|A_{4,5}f| \leqslant \max|f|$ and the readily checked formula (3), and for A_6 from the possibility of integrating series (4) term by term. The norm of operator A_6 is bounded

$$\max_{D_2}|A_6 f| \leqslant \frac{\pi}{2}\max_{D_2}|f - A_5 f| \leqslant \pi\max_{D_2}|f|$$

which directly follows from the Northcott inequality [5], which was proved for real functions. Extension to the complex case is trivial.

Example 1. The expression for the polar angle θ in terms of eccentricity angle φ is elementary

$$\theta = \varphi + i\ln\frac{1 - re^{i\varphi}}{1 - re^{-i\varphi}}, \quad r = \frac{\nu}{1 + \sqrt{1 - \nu^2}}$$

where ν is the eccentricity. It is obvious that $\theta - \varphi - 1$ is a d'Alambert function for whose Fourier expansion coefficients from (9) we have

$$\overline{\lim}\sqrt[|n|]{|c_n(r)|} \leqslant |r|$$

In reality (see [6]) we have here an identity with respect to r

Example 2. The Cartesian coordinates X_1, X_2, X_3 in elliptic motion can be expressed in terms of Keplerian coordinates as follows:

$$X_1 + iX_2 = ae^{i\lambda}e^{i(\varphi-l)}\left[(1-r_1^2)(1-r_3^2) + r_1^2r_3^2e^{2i(\varphi_1-\varphi_2)}\right] +$$
$$ae^{-i\lambda}e^{-i(\varphi-l)}\left[(1-r_1^2)r_3^2e^{2i\varphi_2} + r_1^2(1-r_3^2)e^{2i\varphi_1}\right] -$$
$$a(1-r_1^2)r_2e^{i\varphi_2} - ar_1^2r_2e^{i(2\varphi_1-\varphi_2)}$$
$$iX_3 = ar_1\sqrt{1-r_1^2}\,\{e^{i\lambda}e^{i(\varphi-l)}\left[(1-r_3^2)e^{-i\varphi_1} - r_3^2e^{i(\varphi_1-2\varphi_2)}\right] -$$
$$e^{-i\lambda}e^{-i(\varphi-l)}\left[(1-r_3^2)e^{i\varphi_1} - r_3^2e^{-i(\varphi_1-2\varphi_2)}\right] +$$
$$r_2\left[e^{i(\varphi_1-\varphi_2)} - e^{-i(\varphi_1-\varphi_2)}\right]\}$$
$$(r_3 = r_2\left[2\left(1+\sqrt{1-r_2^2}\right)\right]^{-1/2}),$$

where and in what follows a is the major semiaxis, λ is the mean longitude, φ and l are the eccentric and mean polar angles, r_1 is the sine of half-obliquity, r_2 is the eccentricity, φ_1 is the longitude of the ascending node, and φ_2 is the longitude of the pericenter. We consider $a, \lambda, r_1, \varphi_1, r_2, \varphi_2$ to be independent elements. Obviously r_3^2 is a d'Alambert 1- function of r_2, and $|r_3|^2 \leqslant |r_2|^2/2$.

It was proved in [7] that for any fixed $b \geqslant 1$ the function $\varphi - l$ of three variables λ, r_2, and φ_2 is holomorphic when

$$|r_2|\, e^{|\mathrm{Im}\,\varphi_2|} < 1/\mathrm{ch}\, b, \quad |\mathrm{Im}\,\lambda| \leqslant b - 1$$

and in that region $|\mathrm{Im}(\varphi - l)| < 1$.

Thus for any $a \in \mathbf{C}$ and λ in the band $|\mathrm{Im}\,\lambda| \leqslant b-1$ the coordinates X_1, X_2, X_3 are bounded holomorphic 1- and $(1/\mathrm{ch}\, b)$- d'Alambert function with respect to r_1 and φ_1 and r_2 and φ_2, respectively.

In the smaller region that does not contain collision points the Hamiltonian and the perturbing function of the π-body problem are d'Alambert functions with respect to the pairs (r_{1k}, φ_{1k}) and (r_{2k}, φ_{2k}), where k is the number of the planet.

Note that holomorphy with respect to eccentricity and real remaining variables is guaranteed in circle $|r_2| < 1/\mathrm{ch}\,1 = 0.648054$, which lies inside the Laplace circle $|r_2| < 0.662743$. According to [7] the Laplace limit is reached only when the structure of the specified region of d'Alambert functions is complex; to restrict it to the Reinhart region (see [3]) is not possible.

REFERENCES

1. Poincaré, A., Course of Celestial Mechanics. Moscow, "Nauka", 1965.

2. Arnol'd, V.I., Small denominators and the problem of motion stability in classical celestial mechanics. Uspekhi Matem. Nauk, Vol. 18, №6, 1963.

3. Shabat, B.V., Introduction to Complex Analysis. Moscow, "Nauka", 1969.

4. Henrard, J., Virtual singularities in the artificial satellite theory. Celest. Mech., Vol. 10, №4, 1974.

5. Northcott, D.G., Some inequalities between periodic functions and their derivatives. J. London Math. Soc., Vol. 14, pt. 3, №55, 1939.

6. Subbotin, M.F., Introduction to Theoretical Astronomy. Moscow, "Nauka", 1968.

7. Kholshevnikov, K.V., On the behaviour of fundamental functions of celes-

tial mechanics in a complex region. Collection: Dynamics and Evolution of Stellar Systems. Ser. Problems Universe Exploration, no. 4. Moscow-Leningrad, 1975.

Translated by J.J. D.